An interval nonlinear programming approach for solving a class of unconstrained nonlinear fuzzy optimization problems

Abbas Akrami, Majid Erfanian

Department of Science, School of Mathematical Sciences, University of Zabol, Zabol, Iran
Department of Science, School of Mathematical Sciences, University of Zabol, Zabol, Iran

Abstract

In this paper, an interval programming method is presented for solving unconstrained nonlinear fuzzy optimization problems where all the coefficients of the objective function are triangular fuzzy numbers. First, we convert unconstrained nonlinear fuzzy optimization problems unconstrained interval nonlinear programming problem by α-cuts. Then by solving the unconstrained interval nonlinear programming model, the optimal solution of the main problem will be obtained. To illustrate the proposed method numerical examples are solved and the obtained results are discussed.

Keywords: Fuzzy Optimization, Interval Optimization, Fuzzy Numbers.

1. Introduction

In traditional optimization problems, the coefficients of the problems are evermore treated as deterministic values. However, uncertainty always exits in practical engineering problems. In order to deal with the uncertain programming, fuzzy and stochastic approaches are generally used to describe the imprecise characteristics. In stochastic programming (e.g. [4] (1959); [11] (1982); [16] (2003); [6] (2005)) the uncertain coefficients are regarded as random variables and their probability distributions are assumed to be known. In fuzzy programming (e.g. [25] (1986); [7] (1989); [17] (1989); [15] (2001)) the constraints and objective function are viewed as fuzzy sets and their membership functions need to be known. In these methods, the membership functions and probability distributions play important roles. However, it is sometimes difficult to specify an appropriate membership function or accurate probability distribution in an uncertain environment.

Newly, the interval analysis method was developed to model the uncertainty in uncertain optimization problems, in which the bounds of the uncertain coefficients are only required, not necessarily knowing the probability distributions or membership functions. Many researchers (Tanaka et al. (1984), Rommelfanger (1989), Chanas and Kuchta (1996a,b), Tong (1994), Liu and Da (1999), Sengupta et al. (2001), Zhang et al. (1999) and etc.) studied the linear interval number programming problems. Nevertheless, for most of the engineering problems, the objective function are nonlinear, and they are always obtained through numerical algorithms such as finite element method (FEM) instead of the explicit expressions. The reference (Ma, 2002), seems the first publication on nonlinear interval number programming (NINP). In this reference, a deterministic optimization method is used to obtain the interval of the nonlinear objective function. As a result, an effective method still has not been developed to deal with the general NINP problem in which there exit not only uncertain nonlinear objective function but also uncertain...
nonlinear constraints, so far.

Fuzzy set theory has been applied to many disciplines such as control theory and operation research, mathematical modeling and industrial applications. Tanaka, et al [28], first proposed the concept of fuzzy optimization on general level. Zimmerman [33] proposed the first formatting of fuzzy linear programming. An optimal solution of fuzzy nonlinear programming problems introduced by A. Kumar and J. Kaur [13] and B.Kheirfam [12]. In their works, they have taken all coefficients and decision variables to be fuzzy numbers. K.David Jamison and Weldon Lodwick in [10] are investigated minimizing unconstrained fuzzy functions. In this work, they introduce a definition for a fuzzy function as a possibility distribution over the set of all functions with certain properties. Then, they examine some of the implications of this definition. In addition, a computational method to solve unconstrained fuzzy optimization problems for a class of fuzzy functions has developed by them in [2]. In [1,21] nonlinear unconstrained fuzzy optimization problems has considered and using the concept of convexity and Hukuhara differentiability of fuzzy-valued functions, the necessary and sufficient optimality conditions are derived.

In this paper, we focus to solve unconstrained nonlinear fuzzy optimization problems (UNFOP). We take all coefficients of the objective function to be triangular fuzzy numbers. We convert the UNFOP into a crisp form with using the α-cuts and the obtained problems will be numerically solved. This paper is organized as follows: in section 2, some basic definitions and arithmetic operations of triangular fuzzy numbers and intervals are reviewed. In section 3, convert UNFOP to UINP is discussed by α-cuts technique. In section 4, to demonstrate the effectiveness of the proposed method, some examples are solved. The conclusion appears in section 5.

2 Preliminaries

**Definition 2.1** Let \( l = \{K: K = [a, b], a, b \in R \} \) and let \( A, B \in l \) then the interval arithmetic operations are defined by
\[
A * B = \{\alpha + \beta; 0 \leq \alpha, \beta \leq 1\},
\]
where \( \ast \in \{ +, -, \times, / \} \). (Note that: \( l \) is undefined when \( 0 \in B \).

Letting \( A = [a, b] \) and \( B = [c, d] \) it can be shown that it is equivalent to
\[
A + B = [a + c, b + d], \quad (\text{Minkowski addition})
\]
\[
A - B = [a - c, b - d], \quad (\text{Minkowski difference})
\]
\[
A.B = [a.b, c . d] = [\min(ac, ad, bc, bd), \max(ac, ad, bc, bd)],
\]
\[
A = [a, b], \quad \frac{1}{a} \quad \text{if} \quad 0 \notin [a, b],
\]
\[
kA = \{ka: a \in A\}. \quad (\text{Scalar multiplication})
\]

This means that each interval operation \( \ast \in \{ +, -, \times, / \} \) is reduced to real operations and comparisons.

Now, some definitions and notations of fuzzy set theory are reviewed.

**Definition 2.2** [21] Let \( R \) be the set of real numbers and \( \tilde{a}: R \rightarrow [0,1] \) be a fuzzy set. We say that \( \tilde{a} \) is a fuzzy number if it satisfies the following properties:

(i) \( \tilde{a} \) is normal, that is, there exists \( x_0 \in R \) such that \( \tilde{a}(x_0) = 1 \).

(ii) \( \tilde{a} \) is fuzzy convex, that is,
\[
\tilde{a}(tx + (1-t)y) \geq \min\{\tilde{a}(x), \tilde{a}(y)\}; \quad \forall x, y \in R, t \in [0,1]
\]

(iii) \( \tilde{a} \) is upper semi continuous on \( R \), that is, \( \{x | \tilde{a}(x) \geq \alpha \} \) is a closed subset of \( R \) for each \( \alpha \in [0,1] \);

(iv) \( cl\{x \in R | \tilde{a}(x) > 0 \} \) forms a compact set.

\( F(R) \) denotes the set of all fuzzy numbers on \( R \). For all \( \alpha \in (0,1] \), \( \alpha \)-level set \( \tilde{a}_\alpha \) of any \( \tilde{a} \in F(R) \) is defined as \( \tilde{a}_\alpha = \{x \in R | \tilde{a}(x) \geq \alpha \} \). The 0-level set \( \tilde{a}_0 \) is defined as the closure of the set \( \{x \in R | \tilde{a}(x) > 0 \} \). By definition of fuzzy numbers, it is proved that, for any \( \tilde{a} \in F(R) \) and for each \( \alpha \in (0,1] \), \( \tilde{a}_\alpha \) is compact convex subset of \( R \), and we write \( \tilde{a}_\alpha = [\tilde{a}_\alpha^L, \tilde{a}_\alpha^U] \).

\[
\text{Definition 2.1 Let } I = \{K: K = [a, b], a, b \in R \} \text{ and let } A, B \in I \text{ then the interval arithmetic operations are defined by}
\]
\[
A * B = \{\alpha + \beta; 0 \leq \alpha, \beta \leq 1\},
\]
where \( \ast \in \{ +, -, \times, / \} \). (Note that: \( I \) is undefined when \( 0 \in B \).

Letting \( A = [a, b] \) and \( B = [c, d] \) it can be shown that it is equivalent to
\[
A + B = [a + c, b + d], \quad (\text{Minkowski addition})
\]
\[
A - B = [a - c, b - d], \quad (\text{Minkowski difference})
\]
\[
A.B = [a.b, c . d] = [\min(ac, ad, bc, bd), \max(ac, ad, bc, bd)],
\]
\[
A = [a, b], \quad \frac{1}{a} \quad \text{if} \quad 0 \notin [a, b],
\]
\[
kA = \{ka: a \in A\}. \quad (\text{Scalar multiplication})
\]

This means that each interval operation \( \ast \in \{ +, -, \times, / \} \) is reduced to real operations and comparisons.

Now, some definitions and notations of fuzzy set theory are reviewed.

**Definition 2.2** [21] Let \( R \) be the set of real numbers and \( \tilde{a}: R \rightarrow [0,1] \) be a fuzzy set. We say that \( \tilde{a} \) is a fuzzy number if it satisfies the following properties:

(i) \( \tilde{a} \) is normal, that is, there exists \( x_0 \in R \) such that \( \tilde{a}(x_0) = 1 \).

(ii) \( \tilde{a} \) is fuzzy convex, that is,
\[
\tilde{a}(tx + (1-t)y) \geq \min\{\tilde{a}(x), \tilde{a}(y)\}; \quad \forall x, y \in R, t \in [0,1]
\]

(iii) \( \tilde{a} \) is upper semi continuous on \( R \), that is, \( \{x | \tilde{a}(x) \geq \alpha \} \) is a closed subset of \( R \) for each \( \alpha \in [0,1] \);

(iv) \( cl\{x \in R | \tilde{a}(x) > 0 \} \) forms a compact set.

\( F(R) \) denotes the set of all fuzzy numbers on \( R \). For all \( \alpha \in (0,1] \), \( \alpha \)-level set \( \tilde{a}_\alpha \) of any \( \tilde{a} \in F(R) \) is defined as \( \tilde{a}_\alpha = \{x \in R | \tilde{a}(x) \geq \alpha \} \). The 0-level set \( \tilde{a}_0 \) is defined as the closure of the set \( \{x \in R | \tilde{a}(x) > 0 \} \). By definition of fuzzy numbers, it is proved that, for any \( \tilde{a} \in F(R) \) and for each \( \alpha \in (0,1] \), \( \tilde{a}_\alpha \) is compact convex subset of \( R \), and we write \( \tilde{a}_\alpha = [\tilde{a}_\alpha^L, \tilde{a}_\alpha^U] \).
Definition 2.3 [23] According to Zadeh’s extension principle, we have addition and scalar multiplications in fuzzy number space \( F(R) \) by their \( \alpha \)-cuts are as follows:

\[
(\vec{a} \oplus \vec{b})_{\alpha} = [\vec{a}_{\alpha}^L + \vec{b}_{\alpha}^L, \vec{a}_{\alpha}^U + \vec{b}_{\alpha}^U]
\]

\[
(\mu \vec{a})_{\alpha} = [\mu \vec{a}_{\alpha}^L, \mu \vec{a}_{\alpha}^U]
\]

We define difference of two fuzzy numbers by their \( \alpha \)-cuts by using H-difference as follows:

\[
(\vec{a} - \vec{b})_{\alpha} = \bar{\vec{a}} \ominus \bar{\vec{b}},
\]

where \( \bar{\vec{a}}, \bar{\vec{b}} \in F(R), \mu \in R \) and \( \alpha \in [0,1] \).

Proposition 2.4 [21] For \( \vec{a} \in F(R) \), we have

(i) \( \vec{a}_{\alpha}^L \) is bounded left continuous nondecreasing function on \( (0,1] \);

(ii) \( \vec{a}_{\alpha}^U \) is bounded left continuous nonincreasing function on \( (0,1] \);

(iii) \( \vec{a}_{\alpha}^L \) and \( \vec{a}_{\alpha}^U \) are right continuous at \( \alpha = 0 \);

(iv) \( \vec{a}_{\alpha}^L \leq \vec{a}_{\alpha}^U \).

Moreover, if the pair of functions \( \vec{a}_{\alpha}^L \) and \( \vec{a}_{\alpha}^U \) satisfy the conditions (i)-(iv), then there exists a unique \( \vec{a} \in F(R) \) such that \( \vec{a}_{\alpha} = [\vec{a}_{\alpha}^L, \vec{a}_{\alpha}^U] \) for each \( \alpha \in [0,1] \).

We define here a partial order relation on fuzzy number space.

Definition 2.5 [21] For \( \vec{a}, \vec{b} \in F(R) \) and \( \vec{a}_{\alpha} = [\vec{a}_{\alpha}^L, \vec{a}_{\alpha}^U], \vec{b}_{\alpha} = [\vec{b}_{\alpha}^L, \vec{b}_{\alpha}^U] \), are two closed intervals in \( R \), for all \( \alpha \in [0,1] \), we define

(i) \( \vec{a} \leq \vec{b} \iff \vec{a}_{\alpha}^L \leq \vec{b}_{\alpha}^L \) and \( \vec{a}_{\alpha}^U \leq \vec{b}_{\alpha}^U \)

(ii) \( \vec{a} < \vec{b} \) if and only if for all \( \alpha \in [0,1] \):

\[
\begin{align*}
\vec{a}_{\alpha}^L < \vec{b}_{\alpha}^L \quad \text{or} \quad \vec{a}_{\alpha}^U < \vec{b}_{\alpha}^U \\
\vec{a}_{\alpha}^U < \vec{b}_{\alpha}^U \quad \text{or} \quad \vec{a}_{\alpha}^L < \vec{b}_{\alpha}^L
\end{align*}
\]

Definition 2.6 [23] The membership function of a triangular fuzzy number \( \vec{a} \) is defined by

\[
\mu_{\vec{a}}(r) = \begin{cases} 
\frac{r - a}{b - a}, & \text{if } a \leq r \leq b \\
\frac{c - r}{c - b}, & \text{if } b < r \leq c
\end{cases}
\]

which is denoted by \( \vec{a} = (a,b,c) \). The \( \alpha \)-level set of \( \vec{a} \) is then:

\( \vec{a}_{\alpha} = [(1 - \alpha)a + \alpha b, (1 - \alpha)c + \alpha b] \)

Definition 2.7 [21] Let \( V \) be a real vector space and \( F(R) \) be a fuzzy number space. Then a function \( \vec{f} : V \rightarrow F(R) \) is called fuzzy-valued function defined on \( V \). Corresponding to such a function \( \vec{f} \) and \( \alpha \in [0,1] \), we define two real-valued functions \( \vec{f}_L \) and \( \vec{f}_U \) on \( V \) as \( \vec{f}_L(x) = (\vec{f}(x))_L \) and \( \vec{f}_U(x) = (\vec{f}(x))_U \) for all \( x \in V \).

3 Unconstrained interval nonlinear programming

Let \( T \) be an open subset of \( R^n \) and \( \vec{f} \) be a fuzzy-valued function defined on \( T \). Consider the following unconstrained nonlinear fuzzy optimization problem:

\[
\min \vec{f}(X) = \sum_{j=1}^{n} \vec{c}_j \vec{f}_j(X)
\]

S.t. \( X \in T \). \hspace{1cm} (3.1)

where \( \vec{c}_j (j = 1, ..., n) \) are triangular fuzzy numbers and \( \vec{f}_j(X) \) are nonlinear real-valued functions on \( T \).

Definition 3.1 Let \( T \subseteq R^n \) be an open subset of \( R^n \).

(i) A point \( X^0 \in T \) is a locally non-dominated solution of the problem (3.1) if there exists no \( X^0 \neq X^1 \in N_{\varepsilon}(X^0) \cap T \) such that \( \vec{f}(X^1) \prec \vec{f}(X^0) \).

(ii) A point \( X^0 \in T \) is a non-dominated solution of the problem (3.1) if there exists no \( X^0 \neq X^1 \in T \) such that \( \vec{f}(X^1) \prec \vec{f}(X^0) \).
(iii) A point $X^0 \in T$ is a locally weak non-dominated solution of the problem (3.1) if there exists no $(X^0 \neq )X^1 \in N_e(X^0) \cap T$ such that $\tilde{f}(X^1) \leq \tilde{f}(X^0)$. 

(iv) A point $X^0 \in T$ is a weak non-dominated solution of the problem (3.1) if there exists no $(X^0 \neq )X^1 \in T$ such that $\tilde{f}(X^1) \leq \tilde{f}(X^0)$.

Now, we can convert UNFOP to UINP by $\alpha$-cuts technique.

Let $\alpha \in [0,1]$ and $\vec{c}_j = (c_j^1, c_j^2, c_j^3)$ then according to the definition 2.6 and the problem (3.1) we have

$$\tilde{f}_\alpha(X) = \left[\sum_{j=1}^n((c_j^2 - c_j^1)\alpha + c_j^1)f_j(X), \sum_{j=1}^n(c_j^3 - (c_j^2 - c_j^3)\alpha)f_j(X)\right].$$

Therefore, UNFOP (3.1) is converted to UINP problem as

$$\min f(X) = \sum_{j=1}^n[c_j, \vec{c}_j]f_j(X),$$

S.t. $X \in T$. (3.2)

where for $j=1, \ldots, n$:

$c_j = (c_j^2 - c_j^1)\alpha + c_j^1$, $\vec{c}_j = c_j^3 - (c_j^2 - c_j^3)\alpha$.

Therefore, the problems (3.1) and (3.2) are equivalence.

By setting $\alpha = 1$ in the problem (3.2), the following unconstrained nonlinear programming problem will be obtained:

$$\min z' = \sum_{j=1}^n c_j^2f_j(X),$$

S.t. $X \in T$. (3.3)

By setting $\alpha = 0$ in the problem (3.2), the following UINP problem will be obtained:

$$\min z = \sum_{j=1}^n [c_j^1, c_j^3]f_j(X),$$

S.t. $X \in T$. (3.4)

**Theorem 3.2** [19] For UINP Problem (3.4), the best and worst optimum values can be obtained by solving the following problems respectively:

$$\min z = \sum_{j=1}^n c_j' f_j(X),$$

S.t. $X \in T$. (3.5)

$$\min \bar{z} = \sum_{j=1}^n c_j'' f_j(X),$$

S.t. $X \in T$. (3.6)

where

$c_j' = \begin{cases} c_j^1, & f_j(X) \geq 0 \\
 c_j^3, & f_j(X) \leq 0 \end{cases}$ and $c_j'' = \begin{cases} c_j^3, & f_j(X) \geq 0 \\
 c_j^1, & f_j(X) \leq 0 \end{cases}$.

**Theorem 3.3** [19] If the objective function for Problem (3.4) is changed to 'max', the best and worst optimum values can be obtained by solving the following problems respectively:

$$\max \bar{z} = \sum_{j=1}^n c_j'' f_j(X),$$

S.t. $X \in T$. (3.7)

$$\max z = \sum_{j=1}^n c_j' f_j(X),$$

S.t. $X \in T$. (3.8)
where $c'_j$ and $c''_j$ are as defined in theorem 3.1.

**Definition 3.3** If $X'^* = (x'_1, x'_2, ..., x'_n)^T$, $X^* = (x^*_1, x^*_2, ..., x^*_n)^T$ and $X''^* = (\bar{x}'_1, \bar{x}'_2, ..., \bar{x}'_n)^T$ are the non-dominated solutions of the problems (3.3), (3.5) and (3.6) respectively and $z'^*$, $z^*$ and $z''^*$ are the optimum value of the problems (3.3), (3.5) and (3.6) respectively, then the fuzzy non-dominated solution and the fuzzy optimum value of the problem (3.1) are as following respectively:

$$X^* = \{(x^*_1, x^*_2, \bar{x}'_1), (x^*_2, x^*_2, \bar{x}'_2), ..., (x^*_n, x^*_n, \bar{x}'_n)\}^T \text{ and } \bar{f}^* = (z^*, z'^*, z''^*)$$

**Definition 3.4** If $(x^*_i, x''_i, \bar{x}'_i), 1 \leq i \leq n, \bar{x}'_i$ are all triangular fuzzy numbers then $X^*$ is called a strong fuzzy solution. Otherwise, if $\exists i; 1 \leq i \leq n, (x^*_i, x''_i, \bar{x}'_i)$ is not a triangular fuzzy number, then by reordering $(x^*_i, x''_i, \bar{x}'_i)$ such that all elements of $X^*$ remain fuzzy numbers, the solution is called a weak fuzzy solution. Otherwise, the problem (3.1) has not a fuzzy solution.

Therefore, by using theorem 3.2 and definitions 3.3 and 3.4, we can obtain the non-dominated solution of the problem 3.1.

### 4 Numerical examples

In this section, we will explain previous method with presenting several examples. Note that for obtaining the optimal solutions of the nonlinear programming problems, the backtracking approach [20] is used.

**Example 1.** Consider the following unconstrained nonlinear fuzzy programming problem

$$\min z = (1 \otimes x_1^2) \oplus (0.5 \otimes x_2^2) \oplus (3 \otimes x_2) \oplus 4.5, \quad \text{s.t.}$$

$$x_1, x_2 \in R, \quad (5.1)$$

where $1 = (0,1,2), 0.5 = (0.4,0.5,0.6), 3 = (2,3,4) \text{ and } 4.5 = (3.5,4.5,5.5)$ are triangular fuzzy numbers on $R$.

Now, we convert the problem (5.1) to a UINP problem by using $\alpha$-cuts:

$$\min z_\alpha = [\alpha, 2 - \alpha]x_1^2 + [0.1\alpha + 0.4, 0.6 - 0.1\alpha]x_2^2 + [\alpha + 2, 4 - \alpha]x_2 + [\alpha + 3.5, 5.5 - \alpha]$$

**S.t.**

$$x_1, x_2 \in R, \alpha \in [0,1], \quad (5.1a)$$

Setting $\alpha = 1$, the following UINP will be obtained:

$$\min z' = x_1^2 + 0.5x_2^2 + 3x_2 + 4.5 \quad \text{s.t.}$$

$$x_1, x_2 \in R. \quad (5.1b)$$

The optimal solution of this problem is obtained:

$$z'^* = 0, \quad x_1'^* = 0, \quad x_2'^* = -3.$$

with $\alpha = 0$, we have:

$$\min z = [0,2]x_1^2 + [0.4,0.6]x_2^2 + [2,4]x_2 + [3.5, 5.5], \quad \text{s.t.}$$

$$x_1, x_2 \in R.$$

Firstly, let $x_2 \geq 0$ then by use of the theorem 3.2 we have two problems as below:

**I** min $z' = 2x_1^2 + 0.6x_2^2 + 4x_2 + 5.5,$

**S.t.**

$$\quad \bar{x}_1' \in R, \quad \bar{x}_2' \geq 0. \quad (5.1c)$$

The optimal solution of this problem is obtained:

$$\bar{x}_1'^* = 0, \quad \bar{x}_2'^* = 0, \quad z'^* = 5.5.$$

**II** min $z' = 0.4x_2^2 + 2x_2 + 3.5,$

**S.t.**

$$\quad x_1' \in R, x_2' \geq 0. \quad (5.1d)$$

The optimal solution of this problem is:

$$x_1'^* = t \in R, \quad x_2'^* = 0, \quad z'^* = 3.5.$$
Secondary, let $x_2 \leq 0$ by use of the theorem 3.2 we have two problems as below:

(1) \[ \min \bar{z} = 2x_1^n + 0.6x_2^n + 2x_2^n + 5.5, \]
\[ \text{s.t.} \quad \overline{x_1^n} \in R, \overline{x_2^n} \leq 0. \]

The optimal solution of this problem is obtained:
\[ \overline{x_1^n} = 0, \overline{x_2^n} = -1.667, \overline{z^n} = 3.833. \]

(II) \[ \min \bar{z} = 0.4x_1^n + 4x_2^n + 3.5, \]
\[ \text{s.t.} \quad \overline{x_1^n} \in R, \overline{x_2^n} \leq 0. \]

The optimal solution of this problem is:
\[ \overline{x_1^n} = 0, \overline{x_2^n} = -5, \overline{z^n} = -6.5. \]

According to the optimal solutions of the problems (5.1b), (5.1c), (5.1d), (5.1e) and (5.1f), we have
\[ x_1^* = \min \left( x_1^n, x_1^n \right) = 0, \quad \overline{x_1^n} = \max \left( x_1^n, x_1^n \right) = 0, \]
\[ x_2^* = \min \left( x_2^n, x_2^n \right) = -5, \quad \overline{x_2^n} = \max \left( x_2^n, x_2^n \right) = 0, \]
\[ z^* = \min (\bar{z}, \bar{z}^*), \quad \overline{z} = \max (\bar{z}, \bar{z}^*) = 5.5. \]

Therefore, by using definition 3.3, the strong fuzzy solution of the problem (5.1) is:
\[ x_1^* = \overline{x_1^n}, x_2^* = \overline{x_2^n}, \overline{z} = (-5, -3, 0) \]
and the fuzzy optimum value of the objective function is:
\[ \overline{z}^* = (\overline{z}, \overline{z}^*), \overline{z} = (-6.5, 0, 5.5) \]
and we can find its defuzzified value 0 by using center of area method [31]. In addition, defuzzified values of $x_1^*$ and $x_2^*$ are 0 and -3 respectively. Therefore $(x_1^*, x_2^*) = (0, -3)$ is the non-dominated solution of the problem (5.1).

Example 2. Consider the following unconstrained nonlinear fuzzy programming problem:
\[ \min \bar{z} = (1, 2, 4) \boxplus x_1^2 \boxplus (1, 2, 4) \boxplus x_2^2 \boxplus (1, 3, 5), \]
\[ \text{s.t.} \quad x_1, x_2 \in R. \]

Now, we convert the problem (4.2) to an UINP problem by using $\alpha$-cuts:
\[ \min z_{\alpha} = [\alpha + 1, 4 - 2\alpha] x_1^2 + [\alpha + 1, 4 - 2\alpha] x_2^2 + [2\alpha + 1, 5 - 2\alpha], \]
\[ \text{s.t.} \quad x_1, x_2 \in R, \alpha \in [0, 1]. \]

Setting $\alpha = 1$, we obtain the following UINP problem:
\[ \min z' = 2x_1^2 + 2x_2^2 + 3, \]
\[ \text{s.t.} \quad x_1, x_2 \in R. \]

The optimal solution of this problem is:
\[ x_1^* = 0, x_2^* = 0, z^* = 3. \]

Setting $\alpha = 0$, we have:
\[ \min z = [1.4] x_1^2 + [1.4] x_2^2 + [1.5], \]
\[ \text{s.t.} \quad x_1, x_2 \in R. \]

Now, by considering the theorem (3.2), we have two problems as below:

(I) \[ \min \bar{z} = 4x_1^2 + 4x_2^2 + 5, \]
\[ \text{s.t.} \quad \overline{x_1^n}, \overline{x_2^n} \in R. \]

The optimal solution of this problem is:
\[ \overline{x_1^n} = 0, \overline{x_2^n} = 0, \overline{z} = 5. \]

(II) \[ \min \bar{z} = x_1^2 + x_2^2 + 1, \]
\[ \text{s.t.} \]
\[ x_1, x_2 \in \mathbb{R} \].

The optimal solution of this problem is:
\[ x_1^* = 0, x_2^* = 0, z^* = 1 \].

Therefore by using definition 3.3, the strong fuzzy solution of the problem (5.2) is:
\[ x_1^* = \left( x_1^*, x_1', x_1'' \right) = (0,0,0), x_2^* = \left( x_2^*, x_2', x_2'' \right) = (0,0,0) \]
and the fuzzy optimum value of objective function is:
\[ z^* = \left( z^*, z', z'' \right) = (1,3,5) \]
and we can find its defuzzified value 3 by using center of area method [31]. In addition, defuzzified values of \( x_1^* \) and \( x_2^* \) are 0 and 0 respectively. Therefore \( (x_1^*, x_2^*) = (0,0) \) is the non-dominated solution of the problem (5.1).

5 Conclusion

In this paper, an interval programming method has been presented for solving unconstrained nonlinear fuzzy programming problems. Using partial order relation on fuzzy number space, the non-dominated solution of UNFOP defined. This problem was converted into an unconstrained interval nonlinear programming problem by \( \alpha \)-cuts. This form of the problem will be free of the compulsion membership functions for solve. To solve the obtained problems, backtracking line search method [20] has been used. Then according to theorem 3.2, definition 3.3 and definition 3.4 the non-dominated solution and fuzzy optimal value of the main problem have been obtained.

References


