DERIVATIONS ON TRELLISES

Davoud Ebadi, Mohammad Hossein Sattari

Department of pure Mathematics Azarbaijan Shahid Madani University Azar-shahr, Tabriz, Iran
Department of pure Mathematics Azarbaijan Shahid Madani University Azar-shahr, Tabriz, Iran

Abstract

In this paper, we introduce the notion of derivations for a trellis and investigate some related properties of this subject. We give some equivalent conditions under which a derivation is isotone for trellises. Also, we study fixed points and define f-derivation on T and Cartesian derivation on T1 × T2.

1. Introduction

In 1971, H. Skala introduced the notions of pseudo-ordered sets and trellises. Trellises are generalization of lattices by considering sets with a reflexive and antisymmetric, but not necessarily transitive. They are also an extension of lattices by postulating the existence of least upper bounds and greatest lower bounds on pseudo-ordered sets similarly as for partially ordered sets. Any reflexive and antisymmetric binary relation ≤ on a nonempty set T is called a pseudo - ordered on T and (T, ≤) is called a pseudo - order set or poset. Clearly, each partial order is a pseudo-order. A natural example of a pseudo-order on the set of real numbers is obtained be setting x ≤ y if and only if 0y - xa for a fixed positive number a. Two elements x, y are comparable if x ≤ y or y ≤ x. For a subset L of T, the notions of a lower bound, and upper bound, the greatest lower bound (g. l. b), the least upper bound (l. u. b) are defined analogously to the corresponding notions in a posets. Generally the notion of a derivation introduced in algebraic systems such as rings, near-rings, specially in [5]. Some properties of a derivation such as isotonness of a derivation, the set of fixed points of a derivation and relation of derivations with meet-translation as studied by G. Szasz [5]. Derivations on trellises already defined by Shashirekha B. Rai, S. Parameshwara Bhatta in [2] with extra conditions that is made derivations isotone. Many authors investigate other properties of derivations on trellises and other algebraic systems in [1,2,3,5]. Here we introduced the notion of a derivation on trellises with weak conditions. The remainder of this paper is organized as follows. In the second section we review the definitions and important theorems of the trellis. In section 3, an equivalent condition is given for a trellis interms of isotone derivation. Also, we study fixed points and define f-derivation on T and cartesian derivation on T1 × T2.

1. Preliminaries

Definition. Let T be a nonempty set. A trellis is a poset (T, ≤) where any two of whose elements have a (g. l. b) and a (l. u. b). Any posset can be regarded as a diagram (possibly infinite) in which for any pair of distinct points u and v either there is no directed line between u and v, or if there is a directed line from u to v, there is no directed...
line from v to u.

**Definition.** A trellis T is associative if the following conditions hold for all x, y, z ∈ T, (x ∧ y) ∧ z = x ∧ (y ∧ z) or (x ∨ y) ∨ z = x ∨ (y ∨ z).

**Example 2.1.** The poset A = {0, a, b, c, 1} with 0 ≤ a ≤ b ≤ c ≤ 1, 0 ≤ x ≤ 1 for every x ∈ {a, b, c} and 0 ≤ 1 while a and c are noncomparable. Then A is a trellis.

**Example 2.2.** Let A be a set {0, 1, a, b, c, d} with the following pseudo-order:

\[ a \sqsubseteq c \sqsubseteq d, b \sqsubseteq a \sqsubseteq c \sqsubseteq d, 0 \sqsubseteq x \sqsubseteq 1 \text{ for every } x \in \{a, b, c, d\} \text{ and } 0 \sqsubseteq 1. \]

Then A is a trellis but not lattice and associative since, (a ∨ b) ∨ d = d but a ∨ (b ∨ d) = 1.

**NOTE.** Some properties on lattices hold in trellises as following:

- **Proposition 2.8.** An ideal I of a trellis T is a subtrellis of T such that i ⊴ for all x, y ∈ T.

- **Lemma 2.7.** If I is an ideal of a trellis T then I1, I2 are ideals of a trellis I.

- **Theorem 2.3.** A trellis T is modular if the following condition holds for all x, y, z ∈ T:

\[ x ∨ (y ∧ z) = (x ∨ y) ∧ (x ∨ z). \]

- **Theorem 2.6.** In any trellis, the following statements are equivalent:

  i) x ⊴ is transitive
  ii) The operation ∨ and ∧ are associative
  iii) One of the operations ∨ or ∧ is associative.

- **Theorem 2.5.** A set T with two commutative, absorption and part-preserving operations " ∨ ", " ∧ " is a trellis if a ⊴ b is defined as a ∧ b = a or a ∨ b = b.

- **Proposition 2.9.** If A is an ideal of a trellis T and k is a mapping on A satisfying the property k(x ∧ y) = kx ∧ ky, then k(A) is an ideal of A and hence an ideal of T.

- **Proposition 2.10.** If a trellis T satisfies the inequality x ∧ (y ∨ z) ≤ (x ∧ y) ∨ (x ∧ z), then every mapping k on T satisfying k(x ∧ y) = kx ∧ ky, implies k(x ∨ y) = kx ∨ ky.

- **Example 2.1.** The poset A = {0, a, b, c, 1} with 0 ≤ a ≤ b ≤ c ≤ 1, 0 ≤ x ≤ 1 for every x ∈ {a, b, c} and 0 ≤ 1 while a and c are noncomparable. Then A is a trellis.

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3. On Derivations Of Trellises

**Definition.** Let \( (T, \land, \lor) \) be a trellis. A mapping \( d \) of a trellis \( T \) into itself is called a derivation of \( T \) if it satisfies the following condition for all \( x, y \in T \):
\[
d(x \land y) = (d(x) \land y) \lor (x \land d(y)).
\]
We can often \( d(x) \) written as an abbreviation \( dx \).

**Examples 3.1.**
(i) Let \( T \) be a trellis with the least element \( 0 \). We define a function \( d \) by \( dx = 0 \) for all \( x \in T \). Then \( d \) is a derivation on \( T \), which is called the zero derivation.
(ii) Let \( d \) be the identity function on \( T \). Then \( dx = x \).

**Proposition 3.6.** Let \( T \) be a trellis and \( d \) be a derivation on \( T \). Then the following statements hold:

(i) \( d(x) \leq x \)
(ii) \( d(I) \subseteq I \)
(iii) If \( T \) has a greatest element 1 and \( d \) is a derivation on \( T \), then \( dx = (x \land d1) \lor dx \) for all \( x \in T \)
(iv) If \( T \) has a least element 0 and a greatest element 1, then \( d0 = 0 \) and \( d1 = 1 \).

**Example 3.3.** Let \( T = \{0, 1, a, b, c, d\} \) be a trellis with the following pseudo-order:

- \( a \leq b \)
- \( 0 \leq x \leq 1 \) for every \( x \in \{a, b, c, d\} \) and \( 0 \leq 1 \).

Define \( d: T \to T \) by:

- \( d(0) = 0 \)
- \( d(1) = 1 \)
- \( d(a) = a \)
- \( d(b) = b \)
- \( d(c) = d(b) = b \)
- \( d(d) = b \)

\( d \) is not a derivation, because \( d(1) \leq a \) implies \( b \neq a \), but \( d2 \) is a derivation on \( T \).

**Example 3.4.** Let \( T \) be a trellis with a greatest element 1 and \( d \) be a derivation on \( T \). Then \( d1 = 1 \) if and only if \( d \) is an isotone derivation.

**Remark 3.5.** Every mapping \( k \) on trellis \( T \) satisfying \( k(x \lor y) = kx \lor ky \) is an isotone derivation.

**Definition.** Let \( T \) be a trellis with a greatest element 1 and \( d \) be a derivation on \( T \). Then \( d1 = 1 \) if and only if \( d \) is an isotone derivation.

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**Remark 3.6.** Let \( T \) be a trellis with a greatest element 1 and \( d \) be a derivation on \( T \). Then \( d1 = 1 \) if and only if \( d \) is the identity derivation.

**Proposition 3.10.** Let \( T \) be a trellis and \( d \) be a derivation on \( T \). Then the following conditions are equivalent:

(i) \( d \) is the identity derivation
(ii) \( d(x \lor y) = (dx \lor dy) \land (x \lor dy) \)

Proof. The implication (i) \( \Rightarrow \) (ii) is trivial. Taking \( x = y \) with together contraction property of \( d \) implies (ii) \( \Rightarrow \) (i).

**Definition.** Let \( T \) be a trellis and \( d \) be a derivation on \( T \). Then \( dx = x \land d1 \) if and only if \( d \) is an isotone derivation.

**Example 3.11.** Let \( T \) be a trellis with a greatest element 1 and \( d \) be a derivation on \( T \). Then \( dx = x \land d1 \).

**Proposition 3.12.** Let \( T \) be a trellis with a greatest element 1 and \( d \) be a derivation on \( T \). Then \( dx = x \land d1 \).

Proof. Since \( d \) is an isotone, then \( dx \leq d1 \). Note that \( dx \leq x \), we can get \( dx \leq (x \land d1) \), by proposition 3.6 (iii), \( dx = dx \lor (x \land d1) = x \land d1 \).
Remark 3.13. The above proposition illustrates a condition that makes isotone derivation, principle [7].

**Lemma 3.14.** Let $T$ be a trellis and $d: T \to T$ be a derivation. Then $d(dx) = dx$.

**Proof.** We can get, $dx \preceq (dx \land dx) \lor (d(dx) \land x) = d(x \land dx) = d(dx)$ and also by 3.6, $d(dx) \preceq dx$ thus, $d(dx) = dx$.

**Theorem 3.15.** Let $T$ be a trellis and $d: T \to T$ be a derivation satisfying $d(x \lor y) = dx \lor dy$. Then for all $x, y \in T$:

i) $d$ is an isotone derivation

ii) $x \preceq y$ implies $dx = x \land dy$

iii) $dx \land y = dx \land dy$.

**Proof.** (i) Let $x \preceq y$, then $x \lor y = y$ and so $dx \preceq (dx \lor dy) = d(x \lor v) = dy$. (ii) Let $x \preceq y$. Then by (i), $dx \preceq dy$ and $dx \preceq x$. Therefore $dx \preceq x \land dy$. Also $x \land dy \preceq dx$ since, $dx = d(x \land y) = (dx \land v) \lor (x \land dy)$.

(iii) By definition of the derivation, we have $dx \land y \preceq d(x \land y)$ for all $x, y \in L$. Taking $x = dx \land y$ and $y = dx$ in (ii), we have $d(dx \land y) \preceq (dx \land y) \land dx = dx \land y$. Thus $(dx \land y) \lor (dx \land dy) = dx \land y$ implies $dx \land dy \preceq dx \land y$. Since $dx \land y \preceq dy$ thus $d(dx \land y) \preceq dy$. Then by above equality we have $dx \land y \preceq dy$. Also, $dx \land y \preceq dx$. Then we can get, $dx \land y \preceq dx \land dy$. With attention to above inequalities we have $dx \land y = dx \land dy$.

**Corollary 3.16.** Let $T$ be a trellis and $d: T \to T$ be a derivation satisfying $d(x \lor y) = dx \lor dy$. Then for all $x, y \in T$, $d(x \land y) = dx \land dy$.

**Proof.** We have $d(x \land y) \preceq dx \land dy = dx \land y$, since $d$ is isotone and if $x \land y \preceq x, y$ then $d(x \land y) \preceq dx \land dy$. By definition of the derivation, we can get the inverse relation and so $d(x \land y) = dx \land y$ for all $x, y \in L$.

**Corollary 3.17.** If a trellis $T$ satisfies the inequality $x \land (y \lor z) \preceq (x \land y) \lor (x \land z)$ and $d$ be a derivation on $T$, then the following conditions are equivalent:

(i) $d(x \lor y) = dy$

(ii) $d(x \land y) = dx$.

**Corollary 3.18.** Let $T$ be a trellis and $d$ be a derivation on $T$. If $d(x \land y) = dx \land y$, then $d(x \land y) = dx \land dy$.

**Proof.** We have $d(x \land y) = dx \land y$ thus, $d(d(x \land y)) = d((x \land dy))$ then $d(x \land y) = dy \land dx$.

**Corollary 3.19.** Suppose $k$ be a mapping on a trellis $T$ satisfying $k(x \lor y) = ky \lor ky$. Then $k$ is an isotone derivation if and only if $k(x \land y) = kx \land y$.

**Remark 3.20.** This corollary implies to inverse relation 2.7 (i) is established.

**Proposition 3.21.** Let $T$ be a trellis and $d: T \to T$ be a derivation. If $d(x \land y) = dx \land dy$, then $d$ is an isotone derivation.

**Proof.** For all $x, y \in T$. If $x \preceq y$, then $dx = d(x \land y) = dx \land dy \preceq dy$.

**Theorem 3.22.** Let $T$ be a trellis and $d$ be a derivation on $T$. Then the following conditions are equivalent:

i) $d$ is an isotone derivation

ii) $dx \lor dy \preceq d(x \lor y)$.

**Proof.** (i)$\Rightarrow$(ii). By (i), we have $dx \preceq d(x \lor y), dy \preceq d(x \lor y)$, and so $dx \lor dy \preceq d(x \lor y)$.

(ii)$\Rightarrow$(i). Assume that (ii) holds. Let $x \preceq y$. By (ii), $dx \preceq (dx \lor dy)$ thus $dy \lor x = dy$. Thus $dx \preceq dy$.

**Remark 3.23.** Despite lattices, on trellises we can not expect the following statements for an isotone derivation $d$:

1) $d(x \land y) = dx \land dy$

2) $d(x \lor y) = dx \lor dy$.

**Remark 3.24.** It is trivial that every distributive trellis is a modular trellis and every distributive trellis is a associative trellis. Note that by [4, page on 224] every associative trellis is a transitive trellis, and so every distributive trellis is a lattice.

**Definition.** Let $T$ be a trellis and $d$ be a derivation on $T$. Define $Fixd(T) = \{ x \in T | dx = x \}$. By the following proposition we can see that $Fixd(T)$ is down-closed set, that is, $x \in Fixd(T)$ and $y \preceq x$ imply $y \in Fixd(T)$. Moreover if $d$ is isotone, $Fixd(T)$ is an ideal of $T$.

**Proposition 3.25.** Let $T$ be a trellis and $d$ be a derivation on $T$. If $y \preceq x$ and $dx = x$, then $dy = y$.

**Proof.** suppose $x, y$ are arbitrary elements in $L$, $y \preceq x$, then $y = x \land y$. Thus,
\[
\begin{align*}
   dy &= d(x \land y) \\
    &= (dx \land y) \lor (x \land dy) \\
    &= (x \land y) \lor dy \\
    &= y \lor dy \\
    &= y.
\end{align*}
\]

**Theorem 3.26.** Let \( T \) be a trellis and \( d_1 \) and \( d_2 \) be two isotone derivations on \( T \). Then \( d_1 = d_2 \) if and only if \( \text{Fix}d_1(T) = \text{Fix}d_2(T) \).

**Proof.** Trivially, \( d_1 = d_2 \) implies \( \text{Fix}d_1(T) = \text{Fix}d_2(T) \). For the converse, for all \( x \in T \) since \( d_1x \in \text{Fix}d_1(T) = \text{Fix}d_2(T) \) we have \( d_2d_1x = d_1x \). Similarly \( d_1d_2x = d_2x \). On the other hand, isotonness of \( d_1 \) and \( d_2 \) implies that \( d_2d_1x \leq d_2x \) and \( d_1d_2x \leq d_1x \). Also \( d_1d_2x \leq d_2d_1x \), this shows that \( d_2d_1x = d_1d_2x \). It follows that \( d_1x = d_2d_1x = d_1d_2x = d_2x \).

**Definition.** Let \( (A_1, \leq_1) \) and \( (A_2, \leq_2) \) two pseudo-ordered set. By \( A_1 \times A_2 \) we means the set \( A_1 \times A_2 \) with the pseudo-order \( (a_1, a_2) \leq (b_1, b_2) \) if and only if \( a_1 \leq_1 b_1 \) and \( a_2 \leq_2 b_2 \). If \( T_1 \) and \( T_2 \) are trellises, so is \( T_2 \times T_1 \).

**Remark 3.27.** \( (T_1, \wedge, V_1), (T_2, \wedge, V_2) \) are trellises. It consider that for all \( a_1, b_1 \in T_1 \) and \( a_2, b_2 \in T_2, (T_1 \times T_2, V, \wedge) \) with \( (a_1, a_2) \wedge (b_1, b_2) = (a_1 \wedge b_1, a_2 \wedge b_2) \) and \( (a_1, a_2) V (b_1, b_2) = (a_1 V b_1, a_2 V b_2) \) is a trellis.

**Definition.** Suppose \( d_1, d_2 \) are arbitrary derivations on \( T_1, T_2 \) respectively. Define \( d: T_1 \times T_2 \rightarrow T_1 \times T_2 : d(a, b) = (d_1a, d_2b) \) for all \( a \in T_1, b \in T_2 \). Trivially, \( d \) is a derivation and it is called a Cartesian derivation.

**Example 3.28.** Cartesian derivation of identity derivations is the identity derivation and if \( d_1, d_2 \) are isotone derivations then \( d = d_1 \times d_2 \) is an isotone derivation.

**Remark 3.29.** Nonetheless, we can product many derivations in such matter, but there is a derivation on \( T_1 \times T_2 \) that is not a cartesian derivation. For, let \( T_1 = T_2 = T = \{0, 1\} \) be a trellis with \( 0 \leq 1 \) and define \( d: T \times T \rightarrow T \times T \) by \( d(x, y) = (x, y) \) for all \( (x, y) \neq (1, 1) \) and \( d(1, 1) = (1, 0) \). Note that this derivation is not isotone, perhaps isotone derivation on \( T_1 \times T_2 \) are Cartesian derivations.

**Definition.** Let \( T \) be a trellis. A function \( d: T \rightarrow T \) is called an f-derivation on \( T \) if there exists a function \( f: T \rightarrow T \) such that: \( d(x \land y) = (d(x) \land f(y)) \lor (f(x) \land d(y)) \) for all \( x, y \in T \).

**Remark 3.30.** It is obvious that if \( f \) is an identity function then \( d \) is a derivation on \( T \).

**Example 3.31.** Let \( T = \{0, 1, a, b, c\} \) be a trellis with the following pseudo-order: \( a \leq b \leq c \), \( 0 \leq x \leq 1 \) for every \( x \in \{a, b, c\} \) and \( 0 \leq 1 \). Define \( d: T \rightarrow T \) by: \( d0 = 0, da = 0, db = c, dc = c, d1 = a \). Then \( d \) is not a derivation on \( T \) since \( 0 = d(a \land 1) \neq (da \land 1) \lor (a \land d1) = 0 \lor a = a \). If we define \( f \) by: \( f0 = 0, fa = 0, fb = c, fc = c, f1 = 1 \), then \( d \) is an f-derivation on \( T \), for all \( x, y \in T \).

**Proposition 3.32.** Let \( T \) be a trellis and \( d \) be an f-derivation on \( T \). Then the following identities hold for all \( x, y \in T \)

(i) \( dx \leq fx \)

(ii) If \( T \) has a least element \( 0 \), then \( f0 = 0 \) implies \( d0 = 0 \).

**Proof.** (i) For all \( x \in T \), we have \( dx = d(x \land x) = dx \land fx \), thus \( dx \leq fx \). (ii) Since \( dx \leq fx \) for all \( x \in T \), we have \( 0 \leq d0 \leq f0 = 0 \).

**Corollary 3.33.** If \( T \) has a greatest element \( 1 \) and \( d \) is an f-derivation on \( T \), \( f1 = 1 \), then for all \( x \in T \) we have,

(i) If \( fx \leq d1 \), then \( dx = fx \)

(ii) If \( d1 \leq dx \), then \( dx \leq dx \).

**Proof.** (i) we have \( dx = d(x \land d1) = (dx \land f1) \lor (fx \land d1) = dx \lor fx \), then \( fx \leq dx \). From proposition 3.32 (i), we obtain \( dx = fx \). (ii) Since \( dx = d(x \land d1) = (dx \land f1) \lor (fx \land d1) = dx \lor d1 \), we have \( d1 \leq dx \).

**Remark 3.34.** Note that if \( d1 = 1 \), since \( d1 \leq f1 \), we have \( f1 = 1 \). In this case from corollary 3.33 (i), we get \( dx = fx \).

**Definition.** Let \( T \) be a trellis and \( d \) be an f-derivation on \( T \). If \( x \leq y \) implies \( dx \leq dy \), we call \( d \) is an isotone f-derivation.

**Example 3.35.** The example of 3.31, \( d \) is not an isotone f-derivation, since \( c \leq 1 \) but it dose not follow \( dc \leq d1 \), whereas \( f \) is an increasing function on \( T \).

**Corollary 3.36.** Let \( T \) be a trellis and \( d \) be an f-derivation on \( T \). Then for all \( x, y \in T \) we have,

(i) If \( d \) is an isotone f-derivation, then \( dx \lor dy \leq d(x \lor y) \)

(ii) If \( d(x \land y) = dx \land dy \), then \( d \) is an isotone f-derivation.
Proof. (i) We know that \( x \sqsubseteq x \lor y \) and \( y \sqsubseteq x \lor y \). Since \( d \) is isotone, \( dx \sqsubseteq d(x \lor y) \) and \( dy \sqsubseteq d(x \lor y) \). Hence we obtain \( dx \lor dy \sqsubseteq d(x \lor y) \).

(ii) Let \( d(x \land y) = dx \land dy \) and \( x \sqsubseteq y \). Since \( dx = d(x \land y) = dx \land dy \), we get \( dx \sqsubseteq dy \).

References